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On the space of Fredholm operators

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Abstract

We compare various topologies on the space of (possibly unbounded) Fredholm selfadjoint operators and explain their K-theoretic relevance.*

Introduction

The work of Atiyah and Singer on the index of elliptic operators on manifolds has singled out the role of the space of bounded Fredholm operators in topology. It is a classifying space for a very useful functor, the topological K-theory. This means that a continuous family $(L_x)_{x\in X}$ of elliptic pseudo differential operators parameterized by a compact CW-complex X naturally defines an element in the group $K(X)$, the index of the family.

In most examples, the elliptic operators are not bounded operators and thus the notion of continuity has to be defined carefully. The operator theorists have come up with a quick fix. The family $x \mapsto L_x$ of Fredholm operators is called *Riesz continuous* if and only if the families of *bounded operators*

$$x \mapsto L_x(1 + L_x^* L_x)^{-1/2}, \;\; x \mapsto L_x^*(1 + L_x L_x^*)^{-1/2}$$

are continuous with respect to the operator norm. In concrete applications this approach can be a nuisance. For example, consider as in [6] a *Floer family* of *elliptic* boundary value problems (parameterized by $s \in S^1$)

$$u(t) : [0,1] \to \mathbb{C}, \; s \in [0, 2\pi] \;\; \begin{cases} \frac{du(t)}{dt} + a(t)u(t) = 0 & \text{if} \quad t \in (0,1) \\ u(0) \in \mathbb{R} & \text{and} \quad e^{\mathbf{i}s}u(1) \in \mathbb{R} \end{cases} \qquad (BV_s)$$

where $a : [0,1] \to \mathbb{C}$ is a given smooth function. This family ought to be considered continuous but verifying the above definition can be quite demanding. The first technical goal of this paper is to elucidate this continuity issue.

As observed in [1, 3], for K-theoretic purposes it suffices to investigate only (possibly $\mathbb{Z}_2$-graded) selfadjoint operators (super-)commuting with some Clifford algebra action. For example, the space of Fredholm operators on a Hilbert space H can be identified with

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the space of odd, selfadjoint Fredholm operators on the $\mathbb{Z}_2$-graded space $H \oplus H$ via the correspondence

$$L \mapsto \begin{bmatrix} 0 & L^* \\ L & 0 \end{bmatrix}.$$

That is why we will focus exclusively on selfadjoint operators.

In [6] we have argued that in many instances it is much more convenient to look at the graphs of Fredholm selfadjoint operators on a Hilbert space H. If T is such an operator and $\Gamma_T \subset H \oplus H$ is its graph, then Γ_T is a Lagrangian subspace of $H \oplus H$ (with respect to a natural symplectic structure) and moreover, the pair $(H \oplus 0, \Gamma_T)$ is Fredholm. As shown in [5], the space of Fredholm pairs of Lagrangian subspaces is a classifying space for KO^1. (A similar description is valid for all the functors KO^n; see [6].)

A natural question arises. Suppose that two *families of subspaces* determined by the graphs of two families of Fredholm operators are homotopic inside the larger space of Fredholm pairs of Lagrangian subspaces. Can we conclude that the corresponding families of Fredholm *operators* are also homotopic inside the smaller space of operators?

The is the second issue we want to address in this paper. We will consider various topologies on the space of closed, *unbounded* Fredholm operators and analyze when the above graph map $T \mapsto \Gamma_T$ from operators to subspaces is a *homotopy equivalence*. Surprisingly, to answer this question we only need to decide the continuity of Floer type families of boundary value problems. The symplectic reduction technique developed in [6] coupled with the Bott periodicity will take care of the rest.

The paper consists of three sections. In Section 1 we compare two topologies on the space of unbounded Fredholm operators: the gap topology, given by the gap distance between the graphs, and the Riesz topology, described above. In the second section we prove a general criterion (Proposition 2.1) for recognizing when a family of boundary value problems, such as (BV_s), is continuous with respect to the Riesz topology. In the last section we address the connections with K-theory.

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1 Topologies on the space of selfadjoint operators

Let H be a separable real Hilbert space. Denote by $\mathcal{S}$ the space of densely defined, selfadjoint operators on H and by $\mathcal{BS}$ the space of bounded selfadjoint operators $T : H \to H$. Set

$$[\mathcal{BS}] := \{T \in \mathcal{BS};\ \ \|T\| < 1\}.$$

The *Riesz map* is the bijection

$$\Psi : \mathcal{S} \to [\mathcal{BS}],\ \ A \mapsto A(1 + A^2)^{-1/2}.$$

There are two natural metrics on $\mathcal{S}$: the *gap metric*

$$\gamma(A_0, A_1) := \|(\mathbf{i} + A_0)^{-1} - (\mathbf{i} + A_1)^{-1}\| + \|(\mathbf{i} - A_0)^{-1} - (\mathbf{i} - A_1)^{-1}\|,$$

and the *Riesz metric*

$$\rho(A_0, A_1) := \|\Psi(A_0) - \Psi(A_1)\|.$$

Remark 1.1. According to [4, Thm. IV.2.23] we have $\gamma(A_n, A) \to 0$ if and only if

$$\delta(\Gamma_{A_n}, \Gamma_A) \to 0$$

where Γ_T denotes the graph of the linear operator T and δ denotes the gap between two closed subspaces.

Lemma 1.2. *The identity map $(\mathcal{S}, \rho) \to (\mathcal{S}, \gamma)$ is continuous.*

Proof Observe that for every $A \in \mathcal{S}$ we have

$$\frac{1}{\mathbf{i} \pm A} = \frac{A \mp \mathbf{i}}{1 + A^2} = \frac{A}{1 + A^2} \mp \frac{1}{1 + A^2} = \frac{1}{(1 + A^2)^{1/2}} \Psi(A) \mp \mathbf{i} \frac{1}{1 + A^2}$$

and

$$\frac{1}{1 + A^2} = 1 - \Psi(A)^2$$

so that $\|\Psi(A_n) - \Psi(A)\| \to 0$ implies $\|(\mathbf{i} \pm A_n)^{-1} - (\mathbf{i} \pm A)^{-1}\| \to 0$. ■

Denote by $\mathcal{A}$ the C^*-algebra of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that the limits

$$f(\pm\infty) := \lim_{\lambda \to \pm\infty} f(\lambda) \in \mathbb{C}$$

exist. Denote by $\mathcal{A}_0$ the subalgebra defined by the condition

$$f \in \mathcal{A}_0 \Longleftrightarrow f(-\infty) = f(\infty).$$

Define $P_0, P_\pm \in \mathcal{A}_0$ by

$$P_0(\lambda) \equiv 1, \quad P_\pm(\lambda) = (\lambda \pm \mathbf{i})^{-1}.$$

The Stone-Weierstrass approximation theorem shows that the algebra $\mathcal{P}$ generated by $P_0, P_\pm$ is dense in $\mathcal{A}_0$.

The functional calculus for selfadjoint operators show that any $A \in \mathcal{S}$ defines a continuous morphism of C^*-algebras

$$\mathcal{A} \to \mathcal{BS}, \quad f \mapsto f(A).$$

Proposition 1.3. *The following statements are equivalent.*
(i) $\gamma(A_n, A) \to 0$.
(ii) $\|f(A_n) - f(A)\| \to 0$, $\forall f \in \mathcal{A}_0$.

Proof Clearly (ii) $\Longrightarrow$ (i) since $P_\pm \in \mathcal{A}_0$ and

$$\gamma(A_n, A) = \|P_-(A_n) - P_-(A)\| + \|P_+(A_n) - P_+(A)\|.$$

To prove (i) $\Longrightarrow$ (ii) we use an idea in [7, Chap. VIII]. Clearly if $\gamma(A_n, A) \to 0$ then

$$\|P(A_n) - P(A)\| \to 0, \ \ \forall P \in \mathcal{P}.$$

Fix $f \in \mathcal{A}_0$. Since $\mathcal{P}$ is dense in $\mathcal{A}_0$, for every $\varepsilon > 0$ we can find $P \in \mathcal{P}$ such that $\|f - P\| \leq \varepsilon/3$ and then $n(\varepsilon) > 0$ such that, $\forall n \geq n(\varepsilon)$ such that

$$\|P(A_n) - P(A)\| \leq \varepsilon/3.$$

Then, $\forall n \geq n(\varepsilon)$ we have

$$\|f(A_n) - f(A)\| \leq \|f(A_n) - P(A_n)\| + \|P(A_n) - P(A)\| + \|P(A) - f(A)\| \leq \varepsilon. \ \blacksquare$$

Proposition 1.4. *Fix a function $\alpha \in \mathcal{A}$ such that $\alpha(\lambda) \equiv 1$ for $\lambda \gg 1$ and $\alpha(\lambda) \equiv 0$ if $\lambda \ll -1$. Then the following statements are equivalent.*
(i) $\rho(A_n, A) \to 0$
(ii) $\|f(A_n) - f(A)\| \to 0$, $\forall f \in \mathcal{A}$.
(iii) $\gamma(A_n, A) \to 0$ and $\|\alpha(A_n) - \alpha(A)\| \to 0$.

Proof Define $r \in \mathcal{A}$ by

$$r(\lambda) := \frac{\lambda}{(1+\lambda^2)^{1/2}}.$$

The equivalence (i) $\Longleftrightarrow$ (ii) follows exactly as in the proof of Proposition 1.3 using Lemma 1.2 and the fact that the subalgebra spanned by $\mathcal{A}_0$ and r is dense in A. The equivalence (ii) $\Longleftrightarrow$ (iii) relies on Proposition 1.3 and the fact that the algebra spanned by $\mathcal{A}_0$ and α is dense in $\mathcal{A}$. $\blacksquare$

Remark 1.5. (B. Fuglede) The topological spaces $(\mathcal{S}, \rho)$ and $(\mathcal{S}, \gamma)$ are not homeomorphic. Using Proposition 1.4 it is easy to construct an example of a sequence $A_n \xrightarrow{\gamma} A$ such that A_n does not converge to A in the Riesz metric. More precisely consider the space

$$\ell^2 = \Big\{ (x_j)_{n\geq 1}; \ \ x_j \in \mathbb{R}, \ \sum_j x_j^2 < \infty \Big\}$$

with canonical Hilbert basis $\mathbf{e}_1, \mathbf{e}_2, \cdots$. For $n = 0, 1, 2, \cdots$ define

$$A_n : D(A_n) \subset \ell^2 \to \ell^2, \ \ D(A_n) = \Big\{ (x_j)_{j\geq 1} \in \ell^2; \ \ \sum_{j\geq 1} j^2 |x_j|^2 < \infty \Big\}$$

$$A_n \mathbf{e}_j = \begin{cases} j\mathbf{e}_j, & j \neq n \\ -n\mathbf{e}_j, & j = n \end{cases}$$

One can see that

$$\|(\mathbf{i}\pm A_n)^{-1}-(\mathbf{i}\pm A_0)^{-1}\| = \left|\frac{1}{\mathbf{i}+n}-\frac{1}{\mathbf{i}-n}\right| \to 0$$

so that $\gamma(A_n, A_0)\to 0$. On the other hand, if $\alpha\in\mathcal{A}$ is as in Proposition 1.4 then for all sufficiently large n we have

$$\|\alpha(A_n)-\alpha(A_0)\| = 1.$$

We now want to present a simple criterion of ρ-convergence. For any closed densely defined operator we denote by $\mathcal{R}(T)\subset\mathbb{C}$ its resolvent set.

Proposition 1.6. *Suppose $A\in\mathcal{S}$ such that $\mathcal{R}(A)\cap\mathbb{R}\neq\emptyset$. Suppose S_n is a sequence of densely defined symmetric operators satisfying the following conditions.*
(a) $D(A)\subset D(S_n)$.
(b) There exists a sequence of positive numbers $c_n\to 0$ such that

$$\|S_n u\|\leq c_n(\|Au\|+\|u\|),\;\;\forall u\in D(A).$$

Then $A+S_n\in\mathcal{S}$ for all $n\gg 0$ and

$$\rho(A+S_n, A)\to 0.$$

Proof Set $A_n := A+S_n$. According to [4, Thm.IV.2.24] we have

$$\gamma(A_n, A)\to 0$$

while [4, Thm. V.4.1] implies $A+S_n\in\mathcal{S}$ for all sufficiently large n. Let $\beta\in\mathcal{R}(A)\cap\mathbb{R}$ and consider a small closed interval $I=[\beta-\varepsilon,\beta+\varepsilon]$ such that $I\subset\mathcal{R}(A)$. Then, using [4, Thm. VI.5.10] we deduce that for n sufficiently large we have

$$I\subset\mathcal{R}(A_n),\;\;\forall n\gg 0.$$

Pick now a function $\alpha\in\mathcal{A}$ such that $\alpha(\lambda)\equiv 1$ for $\lambda\geq\beta+\varepsilon$ and $\alpha(\lambda)\equiv 0$ for $\lambda\leq\beta-\varepsilon$. Using [4, Thm. VI.5.12] we deduce

$$\|\alpha(A_n)-\alpha(A)\|\to 0.$$

We can now invoke Proposition 1.4 to conclude that $\rho(A_n, A)\to 0$. ■

2 Families of boundary value problems

Consider now as in [6, App. A] the following data.

• A compact, oriented Riemannian manifold (M, g) with boundary $N=\partial M$ such that a tubular neighborhood of $N\hookrightarrow M$ is *isometric* to the cylinder

$$([0,1]\times N, dt^2+g_N)$$

where g_N is a Riemann metric on N and t denotes the outgoing longitudinal coordinate.

• An Euclidean bundle of Clifford modules $E \to M$ with Clifford multiplication

$$\mathbf{c} : T^*M \to \mathrm{End}\,(E).$$

($\mathbf{c}(\alpha)$ is skew-symmetric for any real 1-form α.) Set $E_0 := E\,|_N$

• $D : C^\infty(E) \to C^\infty(E)$ a symmetric Dirac operator with principal symbol $\mathbf{c}$ such that near N it has the form

$$D = J(\partial_t - D_0), \quad J := \mathbf{c}(dt)$$

where $D_0 : C^\infty(E_0) \to C^\infty(E_0)$ is symmetric and independent of t.

• A sequence of symmetric endomorphisms of E independent of t near N such that

$$\|T_n\|_{C^2} \to 0$$

and (near N) the endomorphism JA_n is symmetric. Set $D_n := D + T_n$. Observe that near N D_n has the form

$$D_n := J(\partial_t - D_0 - JT_n).$$

Following [2], we consider the family $\mathcal{P}$ of admissible boundary conditions. It consists of zero order, formally selfadjoint pseudodifferential projectors with the same principal symbol as the Calderon projector of D_0. The symbol of any P in P commutes with the symbol of D_0 so that the commutator $[P, D_0]$ is a zeroth order pseudodifferential operator. We define a metric ν on $\mathcal{P}$ by setting

$$\nu(P, Q) := \Big\|P - Q\Big\| + \Big\|[P - Q, D_0]\Big\|$$

where $\|\bullet\|$ denotes the norm on the space of bounded operators $L^2(E_0) \to L^2(E_0)$.

Suppose now that we are given a projector $P \in \mathcal{P}$ and a sequence $(P_n) \subset \mathcal{P}$. As in [2], we can form the Fredholm selfadjoint operators

$$A_n : D(A_n) \subset L^2(E) \to L^2(E), \quad D(A_n) = \{u \in H^1(E); \;\; P_n u\,|_N = 0\}$$

$$A_n u = D_n u$$

and

$$A : D(A) \subset L^2(E) \to L^2(E), \quad D(A) = \{u \in H^1(E); \;\; Pu\,|_N = 0\}$$

$$Au = Du.$$

Proposition 2.1. *If*

$$\lim_{n\to\infty} \nu(P_n, P) = 0 \tag{2.1}$$

Then

$$\lim_{n\to\infty} \rho(A_n, A) = 0.$$

Proof The proof relies on the following technical result.

Lemma 2.2. *There exists a sequence of bounded, invertible operators $U_n : L^2(E) \to L^2(E)$ such that*
(i) $1 - U_n$ and $1 - U_n^*$ define bounded operators $H^1(E) \to H^1(E)$
(ii) $(U_n - 1), (U_n - 1)^* \to 0$ in the norm topology on the space of bounded operators $H^s(E) \to H^s(E)$, $s = 0, 1$.
(iii) $D(A_n) = U_n^* D(A)$, $\forall n$.

We will prove this lemma after we have finished the proof of Proposition 2.1. Set

$$B_n := U_n A_n U_n^*.$$

Observe that $B_n \in \mathcal{S}$ and $D(B_n) = D(A)$. Moreover

$$\rho(B_n, A_n) = \|\Psi(U_n A_n U_n^*) - \Psi(A_n)\| = \|U_n \Psi(A_n) U_n^* - \Psi(A_n)\|$$

$$= \Big\|(\,(U_n - 1) + 1)\Psi(A_n)(\,(U_n - 1) + 1)^* - \Psi(A_n)\Big\| \leq C\|(U_n - 1)\|_{L^2,L^2} \cdot \|\Psi(A_n)\| \to 0$$

Thus it suffices to show that

$$\rho(B_n, A) \to 0.$$

Observe that for all $u \in D(A)$ we have

$$\|B_n u - Au\| = \|U_n(D + T_n)U_n^* - D\| \leq \|U_n D(U_n^* u - u)\| + \|U_n T_n U_n^* u\|$$

$$\leq \|U_n\|_{L^2,L^2} \|D(U_n^* u - u)\|_{L^2} + C\|T_n\|_{C^2}\|u\|_{L^2} \leq C\Big(\|(U_n^* - 1)u\|_{H^1} + \|T_n\|_{C^2}\|u\|_{L^2}\Big)$$

$$\leq C\Big(\|(U_n^* - 1)\|_{H^1,H^1}\|u\|_{H^1} + \|T_n\|_{C^2}\|u\|_{L^2}\Big)$$

(use the elliptic estimates in [2])

$$\leq C\Big\{\|(U_n^* - 1)\|_{H^1,H^1}(\|Au\|_{L^2} + \|u\|_{L^2}) + \|T_n\|_{C^2}\|u\|_{L^2}\Big\} \leq c_n(\|Au\| + \|u\|)$$

where $c_n \to 0$. Thus, the operator $S_n = B_n - A$ satisfies all the conditions in Proposition 1.6. On the other hand, A has compact resolvent so that $\mathcal{R}(A) \cap \mathbb{R} \neq \emptyset$. We deduce

$$\rho(A, B_n) = \rho(A, A + S_n) \to 0. \quad \blacksquare$$

Proof of Lemma 2.2 Following the constructions in [4, I.§6.4] define

$$\hat{U}_n : L^2(E_0) \to L^2(E_0), \;\; \hat{U}_n = P_nP + (1 - P_n)(1 - P) = 2P_nP - (P_n + P) + 1$$

$$= 2(P + R_n)P - (2P + R_n) + 1 = R_n(2P - 1) + 1.$$

$\hat{U}_n$ is a pseudodifferential operator of order zero with principal symbol 1. Observe that

$$\hat{U}_n^* = PP_n + (1 - P)(1 - P_n)$$

and, as explained in [4, I.§6.4], $\hat{U}_n^*$ is invertible and maps $\ker P$ onto $\ker P_n$. Observe moreover that

$$\|\hat{U}_n - 1\|_{L^2,L^2} \leq \|R_n\|_{L^2,L^2}\|(2P - 1)\|_{L^2,L^2} \to 0. \tag{2.2}$$

Next, observe that

$$[D_0, \hat{U}_n] = [D_0, R_n](2P - 1) + 2R_n[D_0, P]$$

defines a bounded operator $L^2(E_0) \to L^2(E_0)$ and, using (2.1) we deduce

$$\big\| \, [D_0, \hat{U}_n] \, \big\|_{L^2,L^2} \to 0. \tag{2.3}$$

Observe that $\hat{U}_n$ defines in an obvious fashion a bounded operator

$$\hat{U}_n : L^2(E \mid_{[0,1]\times N}) \to L^2(E \mid_{[0,1]\times N})$$

Consider now a smooth increasing function

$$\eta : [0, 1] \to [0, 1]$$

such that $\eta(t) \equiv 0$ for $t < 1/4$ and $\eta(t) \equiv 1$ for $t > 3/4$. We can regard η as a function on the tubular neighborhood of $N \hookrightarrow M$ and then extending it by 0 we can regard it as a smooth function on M. Notice that if u is a section of E then we can regard ηu as a section of $E \mid_{[0,1]\times N}$.

For any section of E smooth *up to the boundary* define

$$U_n u = (1 - \eta)u + \hat{U}_n(\eta u).$$

It is clear that $U_n u$ is smooth up to the boundary. Notice also that there exists a constant $C > 0$ independent of n such that

$$\|U_n u\|_L^2 \leq C\|u\|_{L^2}$$

for any section u smooth up to the boundary. Thus U_n extends to a bounded operator $L^2(E) \to L^2(E)$. Using (2.2) we deduce that

$$\|(U_n - 1)\|_{L^2,L^2} \to 0.$$

We want to show that U_n induces a bounded operator $H^1(E) \to H^1(E)$ and then estimate the norm of $(U_n - 1)$ as a bounded operator $H^1 \to H^1$.

First of all observe that the elliptic estimates for D_0 imply that there exists a positive constant C such that if u is smooth up to the boundary then

$$C^{-1}\|u\|_{H^1([0,1]\times N)} \leq \|\partial_t u\|_{L^2([0,1]\times N)} + \|D_0 u\|_{L^2([0,1]\times N)} \leq C\|u\|_{H^1([0,1]\times N)}$$

Observe that for any section u smooth up to the boundary we have

$$\|U_n u - u\|_{H^1(M)} = \|(1-\eta)u + \hat{U}_n(\eta u) - u\|_{H^1(M)}$$

$$= \|\hat{U}_n(\eta u) - \eta u\|_{H^1(M)} = \|\hat{U}_n(\eta u) - (\eta u)\|_{H^1([0,1]\times N)}$$

$$\leq C\Big(\|\hat{U}_n(\eta u) - (\eta u)\|_{L^2([0,1]\times N)} + \|\partial_t \hat{U}_n(\eta u) - \partial_t(\eta u)\|_{L^2([0,1]\times N)}$$

$$+\|D_0\hat{U}_n(\eta u) - D_0(\eta u)\|_{L^2([0,1]\times N)}\Big) \tag{2.4}$$

Using (2.2) we deduce

$$\|\hat{U}_n(\eta u) - (\eta u)\|_{L^2([0,1]\times N)} \leq c_n\|u\|_{L^2(M)}, \quad c_n \to 0.$$

To estimate the second term in (2.4) notice first that $[\partial_t, \hat{U}_n] = 0$ so that we have

$$\|\partial_t \hat{U}_n(\eta u) - \partial_t(\eta u)\|_{L^2([0,1]\times N)} = \|\hat{U}_n\partial_t(\eta u) - \partial_t(\eta u)\|_{L^2([0,1]\times N)}$$

$$\leq c_n\|\partial_t u\|_{L^2([0,1]\times N)} \leq c_n\|u\|_{H^1(M)}, \quad c_n \to 0.$$

The estimate of the third term in (2.4) requires a bit more work. Observe that

$$D_0\hat{U}_n(\eta u) - D_0(\eta u) = [D_0, \hat{U}_n](\eta u) + \hat{U}_n(D_0\eta u) - D_0(\eta u)$$

$$= \eta\Big([D_0, \hat{U}_n]u + \hat{U}_n(D_0 u) - D_0 u\Big)$$

so that

$$\|D_0\hat{U}_n(\eta u) - D_0(\eta u)\|_{L^2([0,1]\times N)} \leq \|\,[D_0, \hat{U}_n]u\|_{L^2([0,1]\times N)} + \|\hat{U}_n(D_0 u) - D_0 u\|_{L^2([0,1]\times N)}$$

(use (2.2))

$$\leq c_n(\|u\|_{L^2([0,1]\times N)} + \|D_0 u\|_{L^2([0,1]\times N)}) \leq c_n'\|u\|_{H^1(M)}, \quad c_n' \to 0.$$

We have thus found a sequence of positive numbers $c_n \to 0$ such that

$$\|U_n u - u\|_{H^1(M)} \leq c_n\|u\|_{H^1(M)}$$

for every section u smooth up to the boundary. This shows that U_n induces a bounded operator $H^1(M) \to H^1(M)$ and moreover,

$$\|U_n - 1\|_{H^1, H^1} \leq c_n \to 0.$$

One can prove a similar statement concerning U_n^*. Clearly U_n is invertible being so close to 1. Since $\ker P_n = \hat{U}_n^*(\ker P)$ we deduce that $D(A_n) = U_n^* D(A)$. Lemma 2.2 is proved. ■

3 Classifying spaces for K-theory

For clarity purposes we will consider only a special case, that of the functor KO^1. To discuss the other functors KO^n one should use the bigraded Karoubi functors $KO^{p,q}$ as we did in [6]. The proof is only notationally more complicate.

Denote by $\mathcal{F} \subset \mathcal{S}$ (resp. $\mathcal{BF} \subset \mathcal{BS}$, $[\mathcal{BF}] \subset [\mathcal{BS}]$) the subspace of selfadjoint Fredholm operators. $[\mathcal{BF}]$ has three connected components. Two of them $[\mathcal{BF}_\pm]$, are contractible while the third, $[\mathcal{BF}_0]$ is a classifying space for KO^1 (see [1, 2, 3]). We deduce that $(\mathcal{F}, \rho)$ consists of three components

$$\mathcal{F}_\pm := \Psi^{-1}([\mathcal{BF}_\pm]), \;\; \mathcal{F}_0 := \Psi^{-1}([\mathcal{BF}_0])$$

and $(\mathcal{F}_0, \rho)$ is a classifying space for KO^1.

Observe that $H \oplus H$ is a symplectic space with complex structure

$$J = \begin{bmatrix} 0 & -1_H \\ 1_H & 0 \end{bmatrix}$$

and $\Lambda_0 := H \oplus 0$ is a Lagrangian subspace. Define $\mathcal{FL}_0$ the set of Lagrangian subspaces $\Lambda \subset H \oplus H$ such that (Λ_0, Λ) is a Fredholm pair. We topologize $\mathcal{FL}_0$ using the gap distance δ. The space $(\mathcal{FL}_0, \delta)$ is also a classifying space for KO^1 (see [5]).

There is a natural $1-1$ map

$$\Gamma : \mathcal{F}_0 \to \mathcal{FL}_0, \;\; A \mapsto \Gamma_A.$$

According to Lemma 1.2 the map $\Gamma : (\mathcal{F}_0, \rho) \to (\mathcal{FL}_0, \delta)$ is continuous.

Theorem 3.1. *The map*

$$\Gamma : (\mathcal{F}_0, \rho) \to (\mathcal{FL}_0, \delta)$$

is a weak homotopy equivalence.

Proof Fix $A_0 \in \mathcal{F}_0$. We have to show that for every $n > 0$ the induced map

$$\Gamma_* : \pi_n(\mathcal{F}_0, A_0) \to \pi_n(\mathcal{FL}_0, \Gamma_{A_0})$$

is an isomorphism. Observe first that, according to Bott periodicity,

$$\pi_n(\mathcal{FL}_0, \Gamma_{A_0}) \in \mathcal{G} := \Big\{0, \mathbb{Z}, \mathbb{Z}_2\Big\}.$$

The groups in the family $\mathcal{G}$ have a remarkable property. If $G \in \mathcal{G}$ and $\varphi : G \to G$ is a surjective morphism then φ is an isomorphism.

In [6, §5.3], using the symplectic reduction morphism it is shown that the morphism Γ_* is surjective provided the (general) Floer families are ρ-continuous. This continuity was established in Proposition 2.1. Theorem 3.1 is proved. ■

Remark 3.2. In [6] we claimed that the map $\Gamma : (\mathcal{F}_0, \gamma) \to (\mathcal{FL}_0, \delta)$ is a weak homotopy equivalence when in fact the arguments there, detailed in this paper, prove this only for the stronger ρ-topology. This has no effect on the results of [6] but one æsthetical question still lingers. Is the space $\mathcal{F}_0$ equipped with the gap topology a classifying space for KO^1? If the answer is yes (which we continue to belive to be the case) then our claim in [6] is true.

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